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An approximate solution for the dynamic response of wall transfer probes

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Abstract—An approximate solution for the transient Lévêque problem has been developed based on the approximate integral method and the method of weighted residuals. This solution can be used to explore the dynamic response of thermal or mass boundary layers in large amplitude, but non-reversing, unsteady flows. The accuracy of this approximate solution has been proven over a wide range of dimensionless frequencies ($\omega = \Omega(L^2/S_0^2 D)^{1/3}$). The approximate solution forms the basis of a simple method for calculating the instantaneous shear rate from the measurement of instantaneous heat or mass flux at the surface of a wall transfer probe when the quasi-steady state assumption is not valid.

1. INTRODUCTION

HEAT AND mass transfer probes are used extensively to measure wall shear rates in flow systems. For steady flows, it was first demonstrated by Lévêque [1] that the heat (mass) flux across the thermal (concentration) boundary layer is proportional to the one-third power of the local shear rate. According to this relationship, the wall shear rate can be determined by measuring the associated heat (mass) flux at the probe surface. A review of experimental and theoretical problems associated with the application of this technique is given by Hanratty and Campbell [2]. In addition, it should be noted that numerous other works related to this subject have been published [3–6].

When wall transfer probes are applied to unsteady flows in which the variation of shear rate is very slow, a quasi-steady state assumption can be made and the one-third power law can still be used to calculate the instantaneous wall shear rate. However, in many cases, the inertia of the thermal (concentration) boundary layer cannot be neglected and the quasi-steady state assumption is not valid. Hence, the application of wall transfer probes has usually been restricted to low frequency flows where the quasi-steady state assumption is reasonable. To extend the use of wall transfer probes to high frequencies, knowledge of the dynamic behavior of thermal (concentration) boundary layers is required.

Most studies of the dynamic response of wall transfer probes [7–10] assume that the amplitude of shear rate fluctuations is small compared to the time averaged shear rate. With this assumption, the time dependent scalar boundary layer equation can be linearized and solved asymptotically or numerically. Then the transfer function between the heat (mass) flux and the wall shear rate at different frequencies can be derived. The results indicate that the dynamic response is strongly dependent on the dimensionless frequency $\omega\ (= \Omega(L^2/S_0^2 D)^{1/3})$, which is the same as the Strouhal number [3]. When ω is small, the probe response is fast and the quasi-steady state assumption is valid. However, when ω is large, the probe is so slow that the one-third power law cannot apply. For example, when $\omega = 10$, the amplitude of the heat (mass) flux is only 29% of the value predicted by the one-third power law [10].

The above approach is adequate for studying stationary turbulent flows in which the amplitude of fluctuations is small, but is not feasible for flows involving large amplitude oscillations such as physiological pulsatile flows. To study the dynamic behavior of wall transfer probes in large amplitude unsteady flows, Pedley [11] carried out an asymptotic analysis of non-reversing flows and Kaiping [12] presented numerical results for both non-reversing and reversing flows. They concentrated on calculating the heat (mass) fluxing transient given the time-varying wall shear rate, but in fact the inverse problem of calculating the shear rate from the heat (mass) flux is of primary interest for wall shear probes. Considering this point, Mao and Hanratty [13] developed a technique for the inverse problem based on a numerical solution similar to one presented by Kaiping [12]. However, the methods are not straightforward. They demand either assuming a functional form with several unknown parameters for the time dependent shear rate or guessing the initial shear rate and concentration field, and then numerous numerical iterations are required to obtain a good fit to the measurements of heat (mass) flux. Such complicated procedures are not suitable for practical use.

For non-reversing unsteady flows Menendez and Ramaprian [14], using the approximate integral method, were able to reduce the unsteady scalar boundary layer equation to a simple time dependent ordinary differential equation. With this equation, the time-varying shear rate can be calculated if the cor-

NOMENCLATURE

b	time dependence of δ, equation (9)
C	concentration [mol m^{-3}]
D	diffusivity of mass or heat [m^2 s^{-1}]
L	length of the probe [m]
m	exponent of the weighting function
N	heating power or electric current [W, A]
N_c	heat or electric current convected by the fluid
N_L	heat or electric current lost to the substrate
Nu^*	modified Nusselt number defined in equation (12)
Nu_0^*	modified Nusselt number of $O(1)$ in equation (21)
Nu_1^*	modified Nusselt number of $O(\varepsilon)$ in equation (21)
Nu_s^*	modified Nusselt number for $s = 1$
Pe	Péclet number, S_0L^2/D
Pr	Prandtl number, ν/D
R	real part of a complex function
s, S	(dimensionless) wall shear rate [s^{-1}]
S_0	time averaged shear rate [s^{-1}]
Sc	Schmidt number, ν/D
t	time [s]
T	temperature [K]
x, X	(dimensionless) coordinate parallel to the probe surface [m]
x_1	$x^{2/3}$
y, Y	(dimensionless) coordinate perpendicular to the probe surface [m].

Greek symbols

α_1	constants defined in equation (29)
β_1, β_3	constants defined in equation (30)
Γ	gamma function
δ	dimensionless thickness of the thermal (mass) boundary layer
ε	small number for perturbation
η	similarity variable, y/δ
θ	dimensionless temperature or concentration
ν	kinematic viscosity [m^2 s^{-1}]
ξ	similarly variable, $y/x^{1/3}$
τ	dimensionless time
ϕ	b^2
ψ	as defined in equation (32c)
ω, Ω	(dimensionless) angular frequency [s^{-1}].

Subscript

b	bulk condition
w	surface condition.

responding time-varying heat (mass) flux is known. However, due to its approximate nature, the range of validity of this simplified approach, particularly for systems with large ω, still needs to be tested.

In the present paper, we first consider a similar but different problem (transient Lévêque problem), that of determining the dynamic response of wall transfer probes when subjected to a step change in the surface temperature (concentration) under a steady flow condition. This problem has been solved analytically by Soliman and Chambré [15]. Here we develop an approximate solution by combining the integral method and the method of weighted residuals. The accuracy of this approximate solution is tested by comparing its predictions with the analytical solution [15]. By assuming that the same approximation procedures apply to the problem of non-reversing, time varying flows, an approximate solution, which is of the same form as developed in ref. [14], is then derived. Next, this approximate solution is verified for a wide range of ω by comparing its predictions with available results in literature and our own numerical results.

The principal aim of this paper is to show that a simple approximate solution can provide an effective tool for compensating the dynamic lag of wall transfer probes when inertial effects are important. As will be demonstrated, this method is applicable to a wide range of ω and the oscillations of wall shear rate need not be small as long as there is no shear reversal. In addition, a practical approach for determining the value of constants needed for the approximate solution will be discussed.

2. STATEMENT OF THE PROBLEM

We consider a rectangular probe which is embedded within a solid wall flush with the surface and with its long side perpendicular to the direction of mean flow. The temperature or concentration at the probe surface is controlled to maintain a constant value, which is different from the bulk value upstream of the probe.

We assume that the Péclet number, $Pe = S_0L^2/D$, is large enough so that diffusion of heat (mass) in the flow direction can be neglected compared to convection, and that the thermal (concentration) boundary layer is much thinner than the viscous boundary layer (large Prandtl (Pr) or Schmidt (Sc) number), so that the velocity field can be approximated by a linear profile. Thus, the scalar conservation equation can be simplified as:

$$\frac{\partial \theta}{\partial t} + YS(t)\frac{\partial \theta}{\partial X} = D\frac{\partial^2 \theta}{\partial Y^2} \quad (1)$$

where t denotes the time, θ represents the non-dimensional temperature ($(T-T_b)/(T_w-T_b)$) or concentration ($(C-C_b)/(C_w-C_b)$), D is the thermal or mass diffusivity, Y is the distance from the wall, X is the distance from the leading edge of the probe, and $S(t)$ is the time dependent wall shear rate. Equation (1) has also been used by Kaiping [12] and Mao and Hanratty [13] to study the transient behavior of the heat (mass) transfer probe. The underlying assumptions and the justification of using equation (1) were discussed by Kaiping [12]. After introducing the following dimensionless variables:

$$x = \frac{X}{L}, \quad y = \frac{Y}{L}Pe^{1/3}, \quad s = \frac{S}{S_0}, \quad \tau = \frac{t}{L^2/D}Pe^{2/3}$$

where L is the length of the probe in the streamwise direction and S_0 is the time averaged wall shear rate, we obtain the dimensionless governing equation

$$\frac{\partial\theta}{\partial\tau} + ys(\tau)\frac{\partial\theta}{\partial x} = \frac{\partial^2\theta}{\partial y^2} \tag{2}$$

which has to be solved subject to the boundary conditions:

$$\theta(x = 0, y, \tau) = 0 \tag{3a}$$

$$\theta(y = 0, \tau) = 1 \qquad 0 < x \leqslant 1 \tag{3b}$$

$$\theta(x, y \to \infty, \tau) = 0. \tag{3c}$$

For periodic shear rate oscillations, the periodic solution of equation (2) is independent of the initial condition, and its specification is not essential. For convenience we choose the solution of the quasi-steady state equation

$$ys(\tau = 0)\frac{\partial\theta}{\partial x} = \frac{\partial^2\theta}{\partial y^2} \tag{4}$$

as the initial condition as suggested by Kaiping [12]. On the other hand, for the transient Lévêque problem, which describes the transient response to a step change in the surface condition, the initial condition should be

$$\theta(x, y, \tau = 0) = 0. \tag{5}$$

We shall seek an approximate solution for equation (2) subject to the boundary conditions (3) and the initial condition (4) or (5).

3. APPROXIMATE SOLUTION FOR TRANSIENT LÉVÊQUE PROBLEM

In this section an approximate solution for the transient Lévêque problem (equation (2), boundary conditions (3), initial condition (5) with steady shear rate $s = 1$) will be derived. Several approximation procedures will be introduced to reduce the partial differential equation (2) to an ordinary differential equation which will be solved numerically and compared with the analytical solution of Soliman and Chambré [15].

First, an approximate integral method, which is similar to the Von Kármán–Pohlhausen integral method [16], is used to eliminate the y-dependence of equation (2). A temperature (concentration) profile based on the steady state solution (Lévêque problem) is chosen to satisfy the boundary conditions at $y = 0$ and $y \to \infty$:

$$\theta = \frac{1}{\Gamma(4/3)}\int_\eta^\infty e^{-\eta^3}\,d\eta \tag{6}$$

where $\eta = y/\delta(x, \tau)$ and δ represents the thermal (concentration) boundary layer thickness which is dependent on x and τ. Substituting (6) into (2) and integrating from 0 to ∞, we obtain an equation for the boundary layer thickness

$$\Gamma(2/3)\delta\frac{\partial\delta}{\partial\tau} + \delta^2\frac{\partial\delta}{\partial x} = 3 \tag{7}$$

subject to

$$\delta(x = 0, \tau) = 0 \tag{8a}$$

$$\delta(x, \tau = 0) = 0. \tag{8b}$$

Next, the x-dependence is eliminated by the method of weighted residuals [17]. The use of the steady state solution as a trial function and x^m as a weighting function have proven to be very effective in solving transient natural convection problems [17]. Accordingly, we choose

$$\delta = b(\tau)x^{1/3} \tag{9}$$

as the trial function. After substituting (9) into (7), multiplying the resulting equation by the weighting function x^m and integrating from $x = 0$ to 1, it is found that

$$\frac{d\phi}{d\tau} = \frac{m+5/3}{m+1}\frac{2}{3\Gamma(2/3)}(9-\phi^{3/2}) \tag{10}$$

subject to

$$\phi(\tau = 0) = 0 \tag{11}$$

where $\phi = b^2$ and m is a constant which is determined below. A modified Nusselt number is defined as

$$Nu^* = \int_0^1 -\frac{\partial\theta}{\partial y}(y = 0)\,dx = \frac{3}{2\Gamma(4/3)}\frac{1}{\sqrt{\phi}} \tag{12}$$

to represent the overall heat (mass) flux from the probe surface.

The next step is to determine m by matching the approximate solution with the analytical solution in the limiting condition $\tau \to 0$. When $\tau \to 0$ the diffusion term, $\partial^2\theta/\partial y^2$, dominates the convection term, $y\partial\theta/\partial x$ [15, 18], and equation (2) reduces to

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial y^2} \tag{13}$$

which has a well known solution. Solving (13) subject to (3) and then substituting the solution into (12), we obtain

$$Nu^* \sim \frac{1}{\sqrt{(\pi\tau)}} \quad \text{as} \quad \tau \to 0. \tag{14}$$

On the other hand, the approximate solution for Nu^*

as $\tau \to 0$ can be derived by dropping $\phi^{3/2}$ on the right hand side of equation (10), since $\phi \to 0$ as $\tau \to 0$. The simplified version of (10) can then be integrated, and we obtain

$$Nu^* \sim 0.7980\sqrt{\left(\frac{m+1}{m+5/3}\right)}\frac{1}{\sqrt{\tau}} \quad \text{as} \quad \tau \to 0. \quad (15)$$

By equating (14) and (15), the value of m can be determined ($m = -0.3337$) and equation (10) can be rewritten as:

$$\frac{d\phi}{d\tau} = 0.9849(9-\phi^{3/2}). \quad (16)$$

Now with equations (16), (11) and (12), the transient response of the overall heat (mass) flux can be calculated approximately. Equation (16) can be solved analytically, but the analytical form is not convenient. Therefore, we use an ordinary differential equation solver (DVERK) in the IMSL package, which is based on the fifth order Runge–Kutta method, to calculate the numerical results. It is noted that when $\tau \to \infty$, the steady state solution is $Nu^* = 0.8075$, which is identical to Lévêque's solution. The overshoot ratio (Nu^*/Nu^*_s) has been calculated and compared with the analytical solution derived by the Laplace transform method [15]. As shown in Fig. 1, the results indicate that the approximate solution is quite accurate.

4. APPROXIMATE SOLUTION FOR NON-REVERSING UNSTEADY FLOWS

Assuming that the same trial and weighting functions can be applied to the problem of non-reversing unsteady flow, we now can find an approximate solution of equation (2) subject to boundary conditions (3) and initial condition (4) for a periodic shear rate $s(\tau)$. Omitting the details, we directly present the resulting ordinary differential equation

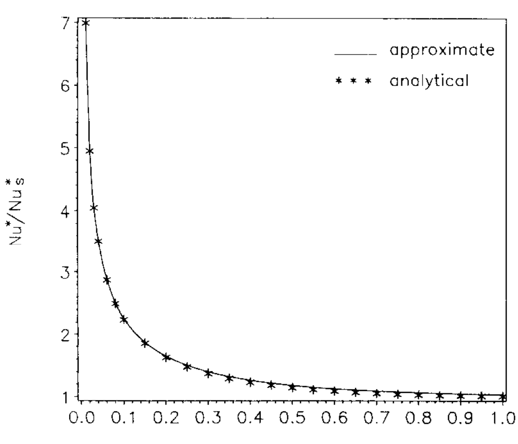

FIG. 1. The approximate (equation (16)) and analytical (Soliman and Chambré [14]) calculations of the response of wall transfer probes when subject to a step change of surface condition (temperature or concentration).

$$\frac{d\phi}{d\tau} = 0.9849(9-s(\tau)\phi^{3/2}) \quad (17)$$

and initial condition

$$\phi(\tau = 0) = (9/s(\tau = 0))^{2/3}. \quad (18)$$

Substituting equation (12) into equation (17), we obtain

$$s(\tau) = 1.8989Nu^{*3} + 1.2089\frac{dNu^*}{d\tau}. \quad (19)$$

Equation (19) has the same form as the approximate solution derived by Menendez and Ramaparian [14] but with different constants. The constants in their approximate solution were determined by comparing its predictions with the numerical solution and adjusting the parameters to obtain a good fit.

The first term on the right hand side of equation (19) corresponds to the quasi-steady state solution (the one-third power law), and the second term represents the inertial effect. With equation (19), the time dependent wall shear rate ($s(\tau)$) can easily be calculated once the overall heat (mass) flux from a probe surface ($Nu^*(\tau)$) has been measured. However, because equation (19) is only an approximate solution derived by using the quasi-steady state solution as trial function, the justification for applying it to a wide range of unsteady flows, especially of large ω, needs to be carefully investigated.

Menendez and Ramaprian [14] have verified that an approximate equation similar to equation (19) can be used to correct the inertia effect for heat transfer probes ($Pr = 7$) up to a fairly high frequency. But for a mass transfer probe, the dynamic response is much slower due to its large Schmidt number (1000–3000 [4]) and the accuracy of (19) still remains unexamined. Because the dimensionless frequency (ω) is proportional to the one-third power of the Prandtl (Schmidt) number, the value of ω for a mass transfer probe could be much larger than that of a heat transfer probe under the same flow condition. For example, when using a mass transfer probe ($Sc = 1370$) to measure wall shear rates of physiological flows, ω could be as high as 12, while it is about 2 for a heat transfer probe ($Pr = 6.9$) [19]. A major goal of the present paper is to prove that equation (19) is still accurate even when ω is large.

5. VERIFICATION OF THE APPROXIMATE SOLUTION

A. *For flows with small shear rate fluctuations*

In this section we will evaluate the performance of equation (19) under the condition that the amplitude of the shear rate oscillation is small compared to the time averaged shear rate. The frequency response of the wall transfer probe under this situation has been studied extensively. A numerical solution covering the whole frequency range has been carried out by Nakoryakov *et al.* [9], and their results are in agreement with the asymptotic solutions of Deslouis *et al.* [10]. The transfer functions given in ref. [9] will be used to verify the accuracy of the approximate solution.

Since the amplitude of fluctuations is small, the time dependent wall shear rate can be represented as

$$s(\tau) = 1+\varepsilon R\{e^{i\omega\tau}\} \tag{20}$$

where $R\{\ \}$ denotes the real part of a complex function and ε represents a small parameter. We seek a solution for the overall flux

$$Nu^*(\tau) = Nu_0^*(1+\varepsilon R\{Nu_1^* e^{i\omega\tau}\}+O(\varepsilon^2)). \tag{21}$$

Substituting equations (20) and (21) into equation (19) and equating the coefficients of like powers of ε on both sides of equation (19), we obtain two algebraic equations for Nu_0^* and Nu_1^*. The results can be expressed as

$$Nu_0^* = Nu_s^* = 0.8075 \tag{22a}$$

$$\frac{Nu_1^*(\omega)}{Nu_1^*(\omega = 0)} = \frac{3.073}{3.073+i\omega} \tag{22b}$$

where Nu_1^* in fact represents the complex amplitude ratio of the dimensionless overall flux to the dimensionless shear rate.

The amplitude and phase of $Nu_1^*(\omega)/Nu_1^*(\omega = 0)$ calculated by equation (22b) and those obtained by Nakoryakov *et al.* [9] are plotted against ω in Figs. 2(a) and (b). The fine agreement over a wide range of ω (0–40) confirms the accuracy of the approximate solution for flows with small shear rate fluctuations.

B. *For low frequency non-reversing unsteady flows*

Pedley [11] carried out a perturbation solution for unsteady flows when the dimensionless frequency ω is small. To examine the performance of the approximate solution (equation (19)) in an unsteady flow with small ω, we compare the approximate solution with Pedley's perturbation solution at $\omega = 0.3$. The result is shown in Fig. 3, and fine agreement is observed. However, since ω is small, the approximate solution does not show much improvement over the quasi-steady state assumption. We expect the advantage of the approximate solution to be more obvious when ω is larger as will be demonstrated in the following section.

C. *For non-reversing unsteady flows*

a. *Description of the numerical method.* Numerical algorithms for both the non-reversing and reversing flows have been reported by Kaiping [12] and Mao and Hanratty [13]. For non-reversing flows, a very simple algorithm, similar to the one used by Lapicque *et al.* [20] to solve the transient Lévêque problem, can be used to generate numerical results to verify the approximate solution. More details are given in the following.

In order to remove the singularity at the leading edge of the probe and to improve the accuracy near it, the coordinate transformations $\xi = y/x^{1/3}$ and

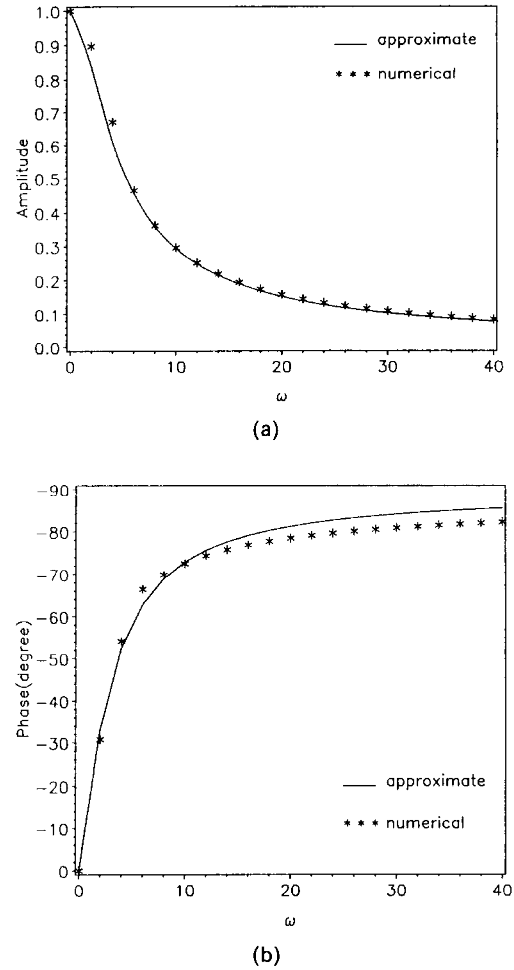

FIG. 2. (a) The approximate (equation (22)) and numerical (Nakoryakov *et al.* [9]) calculations of the amplitude response for wall transfer probes in flows with small shear rate fluctuations. (b) The approximate (equation (22)) and numerical (Nakoryakov *et al.* [9]) calculations of the phase reponse for wall transfer probes in flows with small shear rate fluctuations.

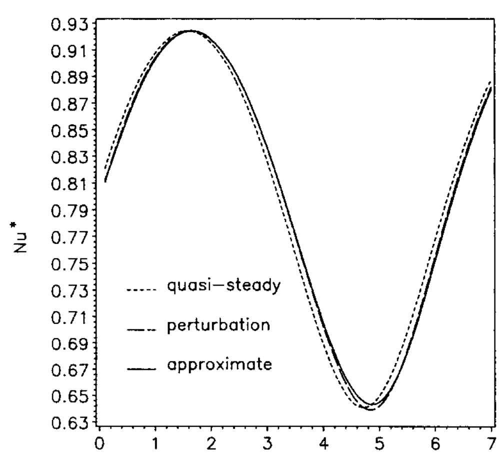

FIG. 3. Comparison between the approximate (equation (19)) and the perturbation solution for $s = 1+0.5\sin(0.3\tau)$.

$x_1 = x^{2/3}$, are introduced and equation (2) can be written as

$$x_1\left(\frac{\partial\theta}{\partial\tau}+\frac{2}{3}\xi s(\tau)\frac{\partial\theta}{\partial x_1}\right)=\frac{\partial^2\theta}{\partial\xi^2}+\frac{1}{3}s(\tau)\xi^2\frac{\partial\theta}{\partial\xi}. \quad (23)$$

The corresponding boundary conditions are

$$\theta(x_1, \xi = 0, \tau) = 1 \quad (24a)$$

$$\theta(x_1, \xi = 6, \tau) = 0 \quad (24b)$$

in which the boundary condition at $\xi \to \infty$ has been replaced by $\xi = 6$ as suggested by the numerical scheme of ref. [13]. The conditions at $x_1 = 0$ and $\tau = 0$ are described by the following equations which can be integrated analytically or numerically:

$$\frac{d^2\theta}{d\xi^2}+\frac{1}{3}s(\tau)\xi^2\frac{d\theta}{d\xi}=0 \quad x_1=0 \quad (25)$$

$$\frac{d^2\theta}{d\xi^2}+\frac{1}{3}s(\tau=0)\xi^2\frac{d\theta}{d\xi}=0 \quad \tau=0. \quad (26)$$

And the overall heat (mass) transfer rate can be calculated by

$$Nu^* = \frac{3}{2}\int_0^1 -\frac{\partial\theta}{\partial\xi}(\xi=0)\,dx_1. \quad (27)$$

The partial derivatives ($\partial/\partial x_1$, $\partial/\partial\xi$ and $\partial^2/\partial\xi^2$) in equation (23) are approximated by second order central differences with uniform grids ($\Delta x_1 = 1/20$ and $\Delta\xi = 1/5$). Then equation (23) can be transformed into a system of time dependent ordinary differential equations (one for each node), which can be integrated by the O.D.E. solver DVERK. To calculate Nu^*, the derivative at the wall is approximated by

$$\frac{\partial\theta}{\partial\xi}(\xi=0)=\frac{-3\theta(\xi=0)+4\theta(\xi=\Delta\xi)-\theta(\xi=2\Delta\xi)}{2\Delta\xi} \quad (28)$$

and the numerical integral of equation (27) is calculated by Simpson's rule.

This numerical scheme was tested for the unsteady shear rate $s(\tau) = 1+0.9\cos(2\tau)$, and the result is in good agreement with Kaiping's numerical solution [12] as shown in Fig. 4.

b. *Verification of the approximate solution.* First, we compare approximate solutions calculated by equation (19) with numerical solutions for the periodic shear rates: $s(\tau) = 1+0.5\sin(\omega\tau)$, $\omega = 5$ and 10. As shown in Figs. 5(a) and (b), the overall flux has a sinusoidal shape which is similar to the quasi-steady state form, but with a smaller amplitude and a phase lag. Next, larger amplitude flows, $s(\tau) = 1+0.9\sin(\omega\tau)$, $\omega = 5$ and 10, are examined. The results depicted in Figs. 6(a) and (b) indicate that the inertia has a greater effect on the low shear rate interval of the response than on the high shear rate interval. Most important of all, the agreement between the approximate and numerical solutions confirms the accuracy of equation (19).

Finally, we explore the dynamic behavior of the wall transfer probe when subjected to a step change

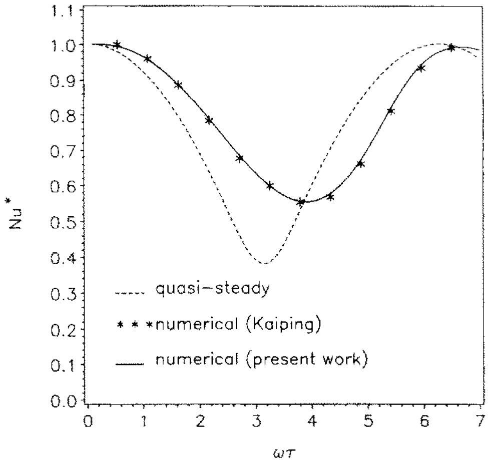

FIG. 4. Comparison between the numerical results of the present work and Kaiping's [12] for $s = 1+0.9\cos(2\tau)$.

(a)
(b)

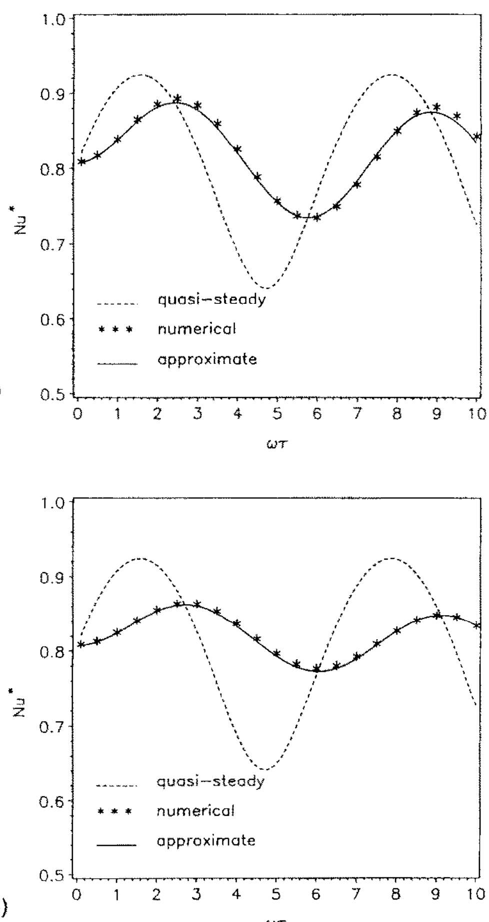

FIG. 5. Comparison between the approximate (equation (19)) and numerical solution for (a) $s = 1+0.5\sin(5\tau)$ and (b) $s = 1+0.5\sin(10\tau)$.

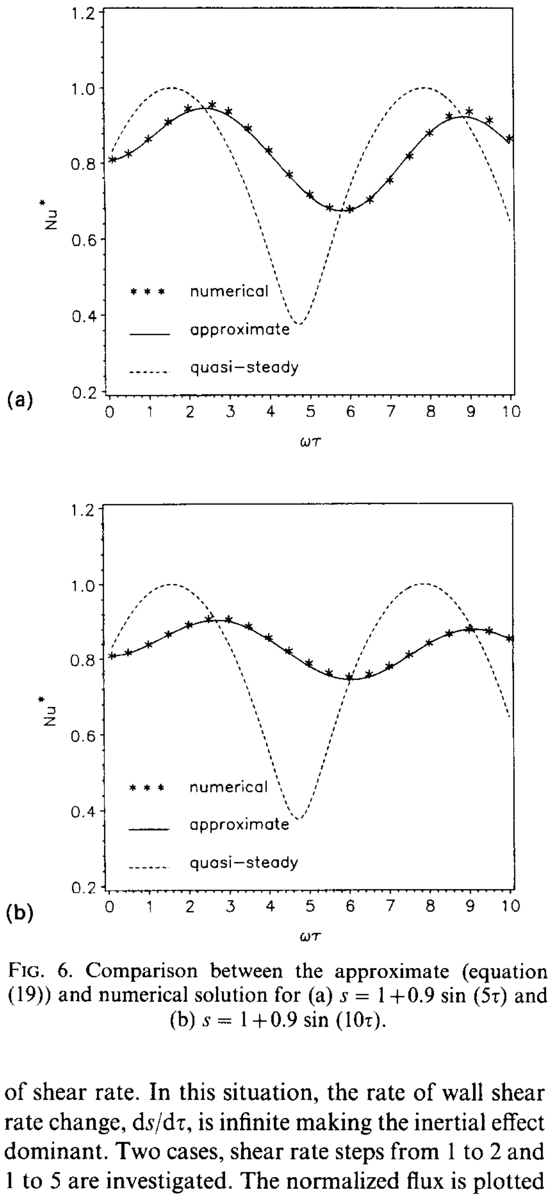

FIG. 6. Comparison between the approximate (equation (19)) and numerical solution for (a) $s = 1+0.9 \sin(5\tau)$ and (b) $s = 1+0.9 \sin(10\tau)$.

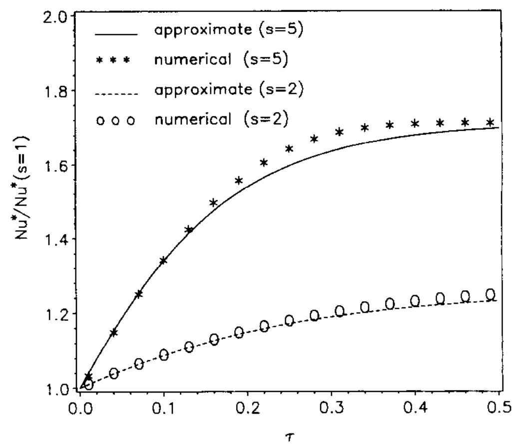

FIG. 7. Comparison between the approximate (equation (19)) and numerical calculations of the response of wall transfer probes when subject to the step changes of wall shear rate: $s = 1 \to 2$ and $s = 1 \to 5$.

of shear rate. In this situation, the rate of wall shear rate change, $\mathrm{d}s/\mathrm{d}\tau$, is infinite making the inertial effect dominant. Two cases, shear rate steps from 1 to 2 and 1 to 5 are investigated. The normalized flux is plotted vs τ in Fig. 7, and a good agreement is again observed.

6. PRACTICAL CALIBRATION

In experimental applications, the overall flux at the probe surface is linearly related to a measurable quantity, N, which could be the electrical power required to maintain a constant probe temperature for heat transfer probes or the electric current in an electrochemical cell for mass transfer probes [2]. This linear relation can be described by

$$Nu^* = \alpha_1 N_C / Pe^{1/3} \tag{29a}$$

$$N_C = N - N_L \tag{29b}$$

where α_1 is a dimensional constant related to the physical properties and operating conditions (T_w, T_b or C_w, C_b), and N_L is a constant which represents the heat loss to the substrate for heat transfer probes and is zero for mass transfer probes since there is no current loss to the substrate. After substituting (29) into (19) we obtain the following dimensional equation:

$$S(t) = \beta_1^3 \left(N_C^3 + \beta_3 \frac{\mathrm{d}N_C}{\mathrm{d}t} \right). \tag{30}$$

Theoretically, if all of the physical properties are known, the operating conditions are specified, and the heat loss to substrate can be estimated, then β_1, β_3 and N_L can be calculated. However, because certain physical properties such as thermal and mass diffusivities are not easy to measure; the performance of the probe is not perfect; and the heat loss is very difficult to estimate; it is usually not feasible to calculate the constants. It is more reliable to determine them by experimental calibration procedures.

For steady flows, equation (30) is reduced to the quasi-steady state solution

$$S^{1/3} = \beta_1 (N - N_L) \tag{31}$$

and it is well known that the constants β_1 and N_L can be obtained by calibrating the probe in steady flow with known wall shear rates [2]. After β_1 and N_L are determined in steady flow, β_3 can be determined by the calibration procedure described below.

Because the electrical circuit response involved in turning a probe on is much faster than the response of the thermal (concentration) boundary layer, turning a probe on is equivalent to imposing a step change in the wall temperature or concentration. This process can be described by the transient Lévêque problem. In fact, good agreement between the solution of the transient Lévêque problem and experimental data for turning on a mass transfer probe has been reported by Lapicque *et al.* [20]. In addition, it is also noted that the approximate solutions for the transient Lévêque problem and for the unsteady non-reversing flow problem are the same (equations (16) and (17)). Consequently, equation (30) can be used to describe the dynamic response of probe turn on, and the constant β_3 can then be determined by matching the approximate solution with the experimental data obtained from a turn on experiment.

Although equation (30) can describe the dynamic response of probe turn on, the corresponding initial condition ($N_c \to \infty$ at $t = 0$), stemming from the discontinuity of wall temperature (concentration) at $t = 0$, is of little practical value. To be able to have a proper initial condition, equation (30) needs to be written in the following form,

$$\frac{d\psi}{dt} = \frac{2}{\beta_1^2\beta_3}(1 - S\psi^{3/2}) \quad (32a)$$

$$\psi(t = 0) = 0 \quad (32b)$$

$$N = \frac{1}{\beta_1\sqrt{\psi}} + N_L \quad (32c)$$

which is similar to equation (16) that describes the time dependence of the thickness of thermal (concentration) boundary layer. To determine the constant β_3 for a probe, a turn on experiment with a steady shear rate needs to be performed. Then the value of β_3 can be determined by adjusting it to match the solution of (32) with the experimental results.

7. SUMMARY AND CONCLUSIONS

The main purpose of this paper was to provide a simple method for calculating the instantaneous shear rate from the corresponding heat or mass flux at a wall transfer probe surface when the quasi-steady state assumption is not valid. To achieve this, an approximate solution for the transient Lévêque problem has been developed based on the approximate integral method and the method of weighted residuals. This solution, which is of the same form as the approximate solution of Menendez and Ramaprian [14], can be used to explore the dynamic response of thermal or mass boundary layers in non-reversing unsteady flows.

For practical use, a dimensional form of the governing ordinary differential equation with three parameters (equation (30)) has been proposed. Two of the three parameters can be determined by the traditional calibration in steady flows at different shear rates. The third parameter can be found by matching the approximate solution with the experimental data obtained by turning a probe on at a steady shear rate. Once the parameters are known, the unsteady shear rate can be calculated from measurements of overall heat (mass) flux using equation (30) with the inertial effects properly compensated.

It should also be noted that there are limitations in the use of equations (19) and (30). As mentioned above, these equations are approximate solutions of equation (1), which itself is a simplified version of the energy (mass) balance equation of the thermal (concentration) boundary layer. The limitations and the justification of equation (1) have been examined by Kaiping [12] and the reader can refer to his work for more details. In addition, the derivation of equation (19) was based on the use of the quasi-steady state solution as the trial function (see Section 3). When ω is small, it is expected that this trial function should give a reasonable approximation. But, when ω is large, the validity needs to be carefully examined. In the present paper, the accuracy of this approximate solution has been tested over a wide range of non-dimensional frequencies by comparison with available solutions in literature and our own finite difference numerical solutions.

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